Linear Algebra with Applications *Third Edition*

Jeanne L. Agnew
Oklahoma State University

Robert C. Knapp
University of Wisconsin, Whitewater

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A zero matrix (Definition 7) acts as an additive identity in the set of matrices of the same size. What matrix is a multiplicative identity?

If the matrix **I** is a multiplicative identity, **IB** = **B**. If **B** is $m \times n$, then **I** must have m columns for multiplication to be defined, and **I** must have m rows for the product **IB** to have m rows. Thus the condition **IB** = **B** forces **I** to be a square matrix.

Example 19

Let **B** be a 2×3 matrix, and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

so that $\mathbf{I}_2\mathbf{B} = \mathbf{B}$ for every 2×3 matrix **B**. The product $\mathbf{BI}_2$ is not defined since **B** has three columns. However,

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Definition 11

Identity Matrix

An $n \times n$ matrix with the property that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$ is called an **identity matrix**, $\mathbf{I}_n$. When the context makes the size of **I** clear, the subscript is omitted.

A zero matrix, like the number 0, also has special properties related to multiplication. For example, if defined, a product in which one factor is a zero matrix always produces a zero matrix.

Example 20

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case $\mathbf{0}_{2\times3}\mathbf{A}_{3\times2} = \mathbf{0}_{2\times2}$. The matrix $\mathbf{A}_{3\times2}\mathbf{0}_{2\times3}$ can also be calculated and the result is the zero matrix, $\mathbf{0}_{3\times3}$. ■

One property of the real number 0 does not carry over into the matrix setting. If the product of two real numbers is 0, we can conclude that at least one of the factors is 0: that is, $ab = 0$ implies either $a = 0$ or $b = 0$. There is no such property in matrix multiplication.

T.S. Blyth and E.F. Robertson

Basic Linear Algebra

Springer

Professor Thomas S. Blyth
Professor Edmund F. Robertson
School of Mathematics and Computer Science, University of St Andrews,
North Haugh, St Andrews KY16 9SS, UK

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Our next result is the multiplicative analogue of Theorem 1.2, but the reader should note that it applies only in the case of square matrices.

Theorem 1.8

There is a unique $n \times n$ matrix M with the property that, for every $n \times n$ matrix A, $AM = A = MA$.

Proof

Consider the $n \times n$ matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

More precisely, if we define the **Kronecker symbol** δ_{ij} by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise,} \end{cases}$$

then we have $M = [\delta_{ij}]_{n \times n}$. If $A = [a_{ij}]_{n \times n}$ then $[AM]_{ij} = \sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij}$, the last equality following from the fact that every term in the summation is 0 except that in which $k = j$, and this term is $a_{ij}1 = a_{ij}$. We deduce, therefore, that $AM = A$. Similarly, we can show that $MA = A$. This then establishes the existence of a matrix M with the stated property.

To show that such a matrix M is unique, suppose that P is also an $n \times n$ matrix such that $AP = A = PA$ for every $n \times n$ matrix A. Then in particular we have $MP = M = PM$. But, by the same property for M, we have $PM = P = MP$. Thus we see that $P = M$. □

Definition

The unique matrix M described in Theorem 1.8 is called the $n \times n$ **identity matrix** and will be denoted by I_n.

Note that I_n has all of its 'diagonal' entries equal to 1 and all other entries 0. This is a special case of the following important type of square matrix.

Definition

A square matrix $D = [d_{ij}]_{n \times n}$ is said to be **diagonal** if $d_{ij} = 0$ whenever $i \neq j$. Less formally, D is diagonal when all the entries off the main diagonal are 0.

EXERCISES

1.19 If A and B are $n \times n$ diagonal matrices prove that so also is AB.

A PRIMER ON
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I. N. HERSTEIN

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DAVID J. WINTER

University of Michigan, Ann Arbor

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Perhaps more interesting are the products

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0.$$

Note several things about the product of matrices:

1. If A, B are in $M_2(\mathbb{R})$, then AB is also in $M_2(\mathbb{R})$.
2. In $M_2(\mathbb{R})$, it is possible that $AB = 0$ with $A \neq 0$ and $B \neq 0$.
3. In $M_2(\mathbb{R})$, it is possible that $AB \neq BA$.

These last two behaviors both run counter to our prior experience with number systems, where we know that

2′. In $\mathbb{R}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.
3′. In $\mathbb{R}$, $ab = ba$ for all a and b.

Here (2′) is, in effect, the *cancellation law of multiplication* for real numbers:

If $ab = 0$ and $a \neq 0$, then $b = 0$.

Thus (2) says that

The cancellation law of multiplication does not hold in $M_2(\mathbb{R})$.

At the same time, cancellation is possible in $M_2(\mathbb{R})$ under certain circumstances, as we observe in Problem 10. Similarly, (3′) says that real numbers *commute* under multiplication. Thus (3) says that

Matrices in $M_2(\mathbb{R})$ do not necessarily commute under multiplication.

Matrices in $M_2(\mathbb{R})$ satisfy the *associative law* that

$$(AB)C = A(BC)$$

as you can see by multiplying out the expressions on both sides of the equation. We leave this as an easy, though tedious exercise.

A matrix that plays a very special role in multiplication is the matrix

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix is called the *identity matrix*, because it has the following properties:

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

Similarly,

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

Thus, multiplying any matrix A by I on either side does not change A at all. In other words, the matrix I in $M_2(\mathbb{R})$ behaves very much like the number 1 does in $\mathbb{R}$ when one multiplies.

For every nonzero real number a we can find a real number, written as $a^{-1} = 1/a$, such that $aa^{-1} = 1$. Is something similar true here in the system $M_2(\mathbb{R})$? The answer is "no." More specifically, we cannot find, for every nonzero matrix A, a matrix A^{-1} such that $AA^{-1} = I$. Consider, for instance, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Can we find a matrix

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

such that $AB = I$? Let's see what is needed. What we require for $AB = I$ is that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix}.$$

This would require that $e = 1$, $f = 0$ and the absurdity that $1 = 0$. So no such B exists for this particular A. Let's try another one, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, where the result is quite different. Again we ask whether we can find a matrix $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ such that $AB = I$. Again, let's see what is needed. What is required in this case is that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e+g & f+h \\ g & h \end{bmatrix}.$$

This requires that $g = 0$, $h = 1$, $e = 1$, $f = -1$, so that the matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

does satisfy $AB = I$. Moreover, this matrix B also satisfies the equation $BA = I$, which you can easily verify.

We have seen that for some matrices A we can find a matrix B such that $AB = BA = I$, and that for some A no such B can be found. We single out these "good" ones in our growing terminology.

Definition. A matrix A is said to be *invertible* if we can find a matrix B such that $AB = BA = I$.

A matrix A which is not invertible is called *singular*. If A is invertible, we claim that the B above is *unique*. What exactly does this mean? It means merely that if $AC = CA = I$ for a (possibly) different matrix C, then $B = C$. To see that $AB = BA = I$ and $AC = CA = I$ imply that $B = C$, just equate $AB = AC$ and cancel A by multiplying each side on the left by B. We leave the details as an exercise. You will have to use the associative law for this (see Problem 3).

If A is invertible and $AB = BA = I$ as above, we call B the *inverse* of A and in analogy to what we do in the real numbers, we write B as A^{-1}. We stress again that *not all* matrices are invertible. In a short while we shall see how the entries of A determine whether or not A is invertible.

We now come to two particular, easy-looking classes of matrices.

Definition. The matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is called a *diagonal matrix*.

Definition. The matrix $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is called a *scalar matrix*.

So a matrix is diagonal if its off-diagonal entries are 0. And it is a scalar matrix if, in addition, the diagonal entries are equal.

Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ be a scalar matrix. Then, if we multiply any

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M. N. Manougian
University of South Florida

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in general, one may *not* cancel the matrix C in an equation $AC = BC$.)

1-3 Elementary Matrices

Objective
Introduce the concept of an elementary matrix of first, second, and third kinds.

This section will be devoted to the introduction of the *identity matrix* and the *elementary matrices* associated with it. These matrices play a significant role in applications discussed later in this chapter.

Let A be an $n \times n$ matrix. We say that A is a **square matrix** of **order** n. We sometimes denote such a matrix by A_n to emphasize the dimension. The entries a_{ii} $(i = 1, 2, \ldots, n)$ are called the elements of the **main diagonal** of the matrix A_n.

Definition 1-6
The identity matrix of order n.

Remember
[illegible]

The matrix I_n defined by

$$I_n = (\delta_{ij})$$

where δ_{ij} is the Kronecker delta, is called the **identity matrix** of order n.

We may describe I_n by saying that all the entries of its main diagonal are 1 while all other entries of I_n are 0. Thus,

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so on.

Let A be a 2×3 matrix. Then we have

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Thus, the effect of multiplying the matrix A by an identity matrix I (of proper order) from the left or from the right is to leave A unchanged.

It follows from the definition of matrix multiplication and the definition of the identity matrix I that we have

$$BI_n = B$$

and

$$I_m B = B$$

for every $m \times n$ matrix B.

In the following examples we shall examine the effect of multiplying a given matrix A (from the left) by special matrices derived from the identity matrix I.

Example 1 Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find EA.

Solution

$$EA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}.$$

A close examination of the resulting matrix shows that one gets the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$ from the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ by interchanging the first and the third rows. Also, one can easily see that interchanging the first and the third rows of I_3 produces the matrix E. Thus, we may think of E as a "row changer" when it multiplies a matrix A from the left. This leads to the following definition.

Definition 1-7 *Elementary matrix of the first kind*

We call E an **elementary matrix of the first kind** if E is obtained from the identity matrix I_n by interchanging any two of its rows.

An important property of an elementary matrix E of the first kind is illustrated in the next example.

Example 2 Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find $E(EA)$.

Solution

$$E(EA) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \right\}$$

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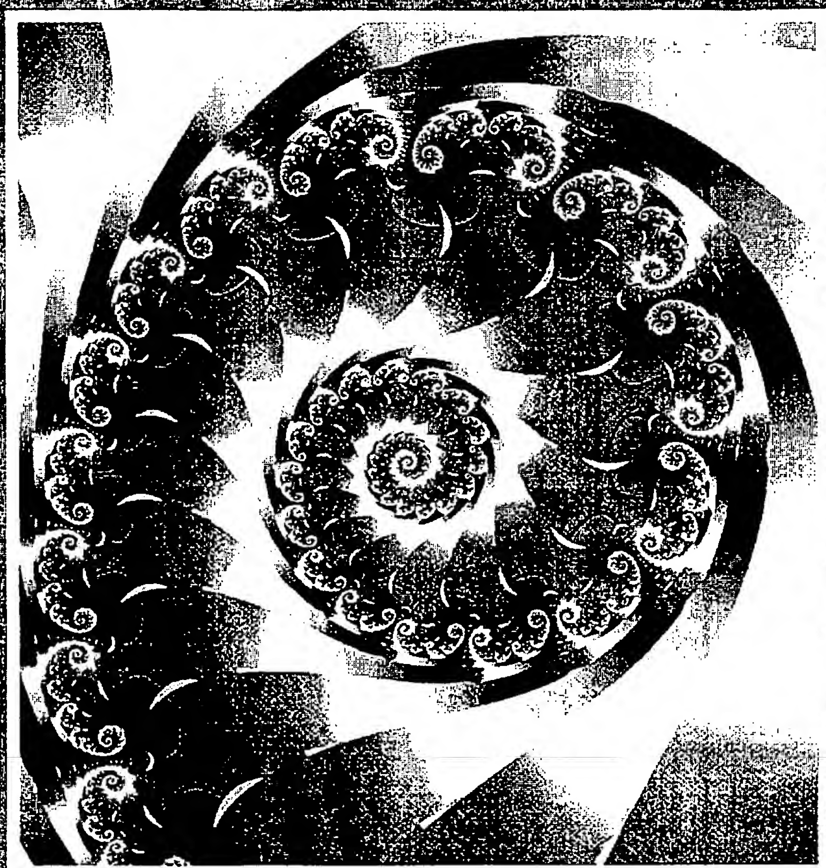

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Dedication:

This work is dedicated to my mother, Deborah
my sister, Karen
and my niece, Chelsea

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3 above.) In general, the matrix I_n—the $n \times n$ diagonal matrix with every diagonal entry equal to 1—is called the **identity matrix** of order n and serves as the multiplicative identity in the set of all $n \times n$ matrices.

Is there a multiplicative identity in the set of all $m \times n$ matrices if $m \neq n$? For any matrix A in $M_{m\times n}(\mathbf{R})$, the matrix I_m is the **left identity** ($I_mA = A$), and I_n is the **right identity** ($AI_n = A$). Thus, unlike the set of $n \times n$ matrices, the set of nonsquare $m \times n$ matrices does not possess a unique *two-sided* identity, because $I_m \neq I_n$ if $m \neq n$.

Example 20: If A is a square matrix, then A^2 denotes the product AA, A^3 denotes the product AAA, and so forth. If A is the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

show that $A^3 = -A$.

The calculation

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

shows that $A^2 = -I$. Multiplying both sides of this equation by A yields $A^3 = -A$, as desired. [Technical note: It can be shown that in a certain precise sense, the collection of matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where a and b are real numbers, is structurally identical to the

Elementary Linear Algebra

EVAR D. NERING

Professor of Mathematics
Arizona State University

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We define δ_{ij} to be

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.16}$$

This useful symbol is known as the ***Kronecker delta***. We shall use it here to construct the n-th order square matrix $I_n = [\delta_{ij}]$. The matrix I_n is called the n-th order ***identity matrix***. It is a diagonal matrix with 1's in the main diagonal. Usually all matrices under consideration are of the same order, or the order of I_n is known from context. Thus, usually we shall simply write I for the identity matrix. If A is an $m \times n$ matrix, then $I_mA = A$ and $AI_n = A$. It is easily seen from the definitions that $\sum_{k=1}^{m} \delta_{ik}a_{kj} = a_{ij}$, and $\sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij}$. In particular, if A is a square matrix and I is an identity of the same order, then $IA = AI = A$.

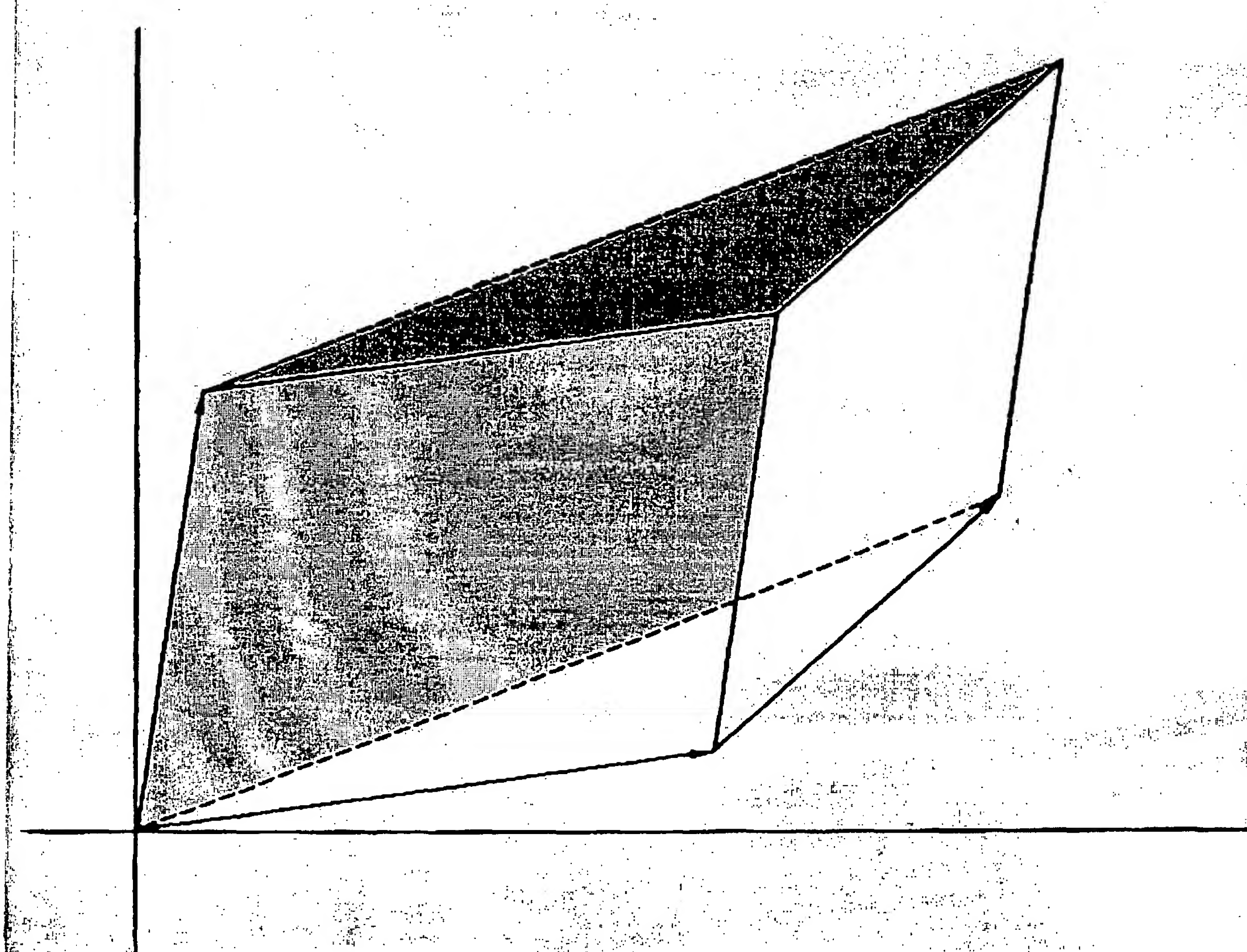
Let A, B, and C be square matrices of the same order such that $AB = I$ and $CA = I$. Then $C = CI = C(AB) = (CA)B = IB = B$. B is called a right inverse of A, and C is called a left inverse of A. We have shown that if A has both a right inverse and a left inverse, then the left inverse and right inverse are identical and that the right inverse is also a left inverse. If A and B are square matrices of the same order such that $AB = BA = I$, then B is called an *inverse* of A. The argument above shows that a matrix cannot have two inverses. We could consider one to be a left inverse and the other to be a right inverse and show they are identical. Thus, we shall speak of *the* inverse of a matrix A and denote it by A^{-1}. Since $AA^{-1} = A^{-1}A = I$, A is also the inverse of A^{-1}. Later (Section 4–4) we will be able to show that if A and B are square matrices of the same order such that $AB = I$, then $BA = I$.

Matrix notation allows us to express the system of equations (1.3) in extremely compact form. Let A be the $m \times n$ matrix whose elements are the coefficients appearing on the left side of (1.3). That is, let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \tag{1.17}$$

MATRICES AND LINEAR ALGEBRA

Hans Schneider
George Phillip Barker



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In $F_{n,n}$ we have a special matrix, called the *identity matrix*.

$$I_n = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix}$$

defined by

$$e_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

For example,

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I_n has the property that for any A in $F_{n,n}$,

$$I_nA = A = AI_n.$$

We note in passing that if B is in $F_{m,n}$,

$$I_mB = B \quad \text{and} \quad BI_n = B,$$

and that for any scalar α,

$$(\alpha I_m)B = \alpha B \quad \text{and} \quad B(\alpha I_n) = \alpha B.$$

Recall that for any number $x \neq 0$, there is a number $y = x^{-1}$, called the *multiplicative inverse* (or reciprocal) of x, such that

$$xx^{-1} = 1.$$

This situation does not hold in $F_{n,n}$, however. For let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$